ON THE EQUILIBRIUM SHAPE AND TENSION OF A FLEXIBLE STRING ACTED UPON BY EXTERNAL FORCES THAT ARE FUNCTIONS OF THE ORIENTATION OF THE STRING IN SPACE

(O FIGURE RAVNOVESIIA I NATIAZHENII GIBKOI NITI POD DEISTVIEM VNESHNIKH SIL, ZAVISIASHCHIKH OT NAPRAVLENIIA NITI V PROSTRANSTVE)

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N.I.ALEKSEEV
(Moscow)

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Particular solutions of the problem about to be considered have been obtained for specific external force fields by Johann Bernoulli (the shape of a sail in a wind), Popov [1], Krylov [2] and Kochin [3].

The problem in general form has been solved only for the case of a planar force field (Minakov [4]). Minakov obtained his solution by assuming that the components of the external force were given along the tangent and normal to the string, and by applying the natural equations of equilibtium.

What follows is a general solution of the problem in Cartesian coordinates with the string situated in a plane and in a three-dimensional space.

1. Let the external force $\mathbf{F}^{\cdot}$ per unit string length be a function of the orientation of the string in space, i.e. of the direction cosines of the string dx/ds , dy/ds , dz/ds and let it be given in terms of its projections F_x , F_y , F_z on the Cartesian coordinate axes.

The equilibrium equations for a flexible inextensible homogeneous string in this case take the form

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right)+F_x\left(\frac{dx}{ds},\ \frac{dy}{ds},\ \frac{dz}{ds}\right)=0 \qquad (x,\ y,\ z)$$

$$\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2+\left(\frac{dz}{ds}\right)^2=1$$

where s is the length of the string, T is the tension in the string, and (x, y, z) is the symbol for the cyclic permutation.

Let us multiply out the derivatives on the left-hand sides of the first three equations of the system, and add to the third equation, multiplied by dz/ds , the first two equations multiplied by dx/ds and dy/ds, respectively. Taking into account the fourth equation of the system and the relation obtained by differentiating the fourth equation with respect to s , we reduce the equilibrium equations to the form

$$\frac{dT}{ds}\frac{dx}{ds}+T\frac{d^2x}{ds^2}+F_x\left(\frac{dx}{ds},\frac{dy}{ds},\frac{dz}{ds}\right)=0$$
$$\frac{dT}{ds}\frac{dy}{ds}+T\frac{d^2y}{ds^2}+F_y\left(\frac{dx}{ds},\frac{dy}{ds},\frac{dz}{ds}\right)=0 \qquad (1.1)$$
$$\frac{dT}{ds}+\frac{dx}{ds}F_x\left(\frac{dx}{ds},\frac{dy}{ds},\frac{dz}{ds}\right)+\frac{dy}{ds}F_y\left(\frac{dx}{ds},\frac{dy}{ds},\frac{dz}{ds}\right)+\frac{dz}{ds}F_z\left(\frac{dx}{ds},\frac{dy}{ds},\frac{dz}{ds}\right)=0$$
$$\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2+\left(\frac{dz}{ds}\right)^2=1$$

Without any significant loss of generality, we may assume that the string nowhere forms a right angle with one of the coordinate axes, e.g. the x-axis. This guarantees that $dx/ds \neq 0$.

We now introduce the new variables u and v, so that, taking into account the fourth equation in system (1.1), we have

$$\frac{dx}{ds}=u, \qquad \frac{dy}{ds}=uv, \qquad \frac{dz}{ds}=\sqrt{1-u^2-u^2v^2} \qquad (1.2)$$

This substitution of variables allows us to express the projections of the external force on the coordinate axes as functions of u and v. The first three equations of system (1.1) become

$$\frac{dT}{ds}u+T\frac{du}{ds}+F_x(u,v)=0 \qquad (1.3)$$

$$\frac{dT}{ds}uv+T\left(\frac{du}{ds}v+u\frac{dv}{ds}\right)+F_y(u,v)=0 \qquad (1.4)$$

$$\frac{dT}{ds}=-uF_x(u,v)-uvF_y(u,v)-\sqrt{1-u^2-u^2v^2}F_z(u,v) \qquad (1.5)$$

Multiplying Equation (1.3) by v and subtracting the result from Equation (1.4) we obtain

$$T\frac{dv}{ds}=\frac{v}{u}F_x(u,v)-\frac{1}{u}F_y(u,v) \qquad (1.6)$$

On the other hand, from Equation (1.3), recalling (1.5), we find that

$$T\frac{du}{ds}=(u^2-1)F_x(u,v)+u^2vF_y(u,v)+u\sqrt{1-u^2-u^2v^2}F_z(u,v) \qquad (1.7)$$

We divide Equation (1.6) by (1.7). This gives us an ordinary first-order differential equation solved for the first derivative that relates the variables u and v

$$\frac{dv}{du}=\frac{vF_x(u,v)-F_y(u,v)}{u(u^2-1)F_x(u,v)+u^3vF_y(u,v)+u^2\sqrt{1-u^2-u^2v^2}F_z(u,v)}$$

Once the function $v(u)$ has been determined by approximate integration of the above equation, finding the five functions $x(u)$, $y(u)$, $z(u)$, $s(u)$, $T(u)$ that constitute the complete parametric solution of the problem becomes a matter of quadratures. Indeed, dividing Equation (1.5) by (1.7), we obtain an equation with separable variables, whose integration yields the following expression for the tension in the string:

$$T(u)=C_1\exp\int\frac{uF_x(u,v)+uvF_y(u,v)+\sqrt{1-u^2-u^2v^2}F_z(u,v)}{(1-u^2)F_x(u,v)-u^2vF_y(u,v)-u\sqrt{1-u^2-u^2v^2}F_z(u,v)}du$$

Next, integrating Equation (1.7), we have

$$s(u)=\int\frac{T(u)\,du}{(u^2-1)F_x(u,v)+u^2vF_y(u,v)+u\sqrt{1-u^2-u^2v^2}F_z(u,v)}+C_2$$

Finally, by integrating Equations (1.2) we obtain the present coordinates of the string as functions of u :

$$x(u) = \int u\, ds(u) + C_3, \qquad y(u) = \int uv\, ds(u) + C_4$$
$$z(u) = \int \sqrt{1 - u^2 - u^2v^2}\, ds(u) + C_5$$

The constants C_1 , C_2 , C_3 , C_4 , C_5 can be determined from the given initial conditions.

2. When the string is situated in a plane, the equilibrium equations for a flexible, inextensible, homogeneous string become

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right) + F_x\left(\frac{dx}{ds}, \frac{dy}{ds}\right) = 0, \qquad \frac{d}{ds}\left(T\frac{dy}{ds}\right) + F_y\left(\frac{dx}{ds}, \frac{dy}{ds}\right) = 0$$
$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1 \tag{2.1}$$

We will attempt to find the tension and equilibrium shape of the string in parametric form, i.e. in the form of four functions of the direction cosine dx/ds :

$$x, y, s, T \mid (dx / ds)$$

We introduce the appropriate notation. Taking into account the third equation of (2.1), we have

$$\frac{dx}{ds} = u, \qquad \frac{dy}{ds} = \sqrt{1 - u^2} \tag{2.2}$$

This substitution allows us to express F_x and F_y as functions of the single variable u .

We multiply out the derivatives on the left-hand sides of the equilibrium equations and add the first equation multiplied by dx/ds to the second equation multiplied by dy/ds . Recalling the third equation of system (2.1) and the relation

$$\frac{dx}{ds}\frac{d^2x}{ds^2} + \frac{dy}{ds}\frac{d^2y}{ds^2} = 0$$

we obtain

$$\frac{dT}{ds} = -uF_x(u) - \sqrt{1 - u^2}F_y(u) \tag{2.3}$$

We note that

$$\frac{d^2x}{ds^2}\frac{dy}{ds} - \frac{d^2y}{ds^2}\frac{dx}{ds} = \frac{d}{ds}\frac{dx}{ds} \Big/ \frac{dy}{ds}$$

Next, we multiply the second equation of system (2.1) by dx/ds and subtract from it the first equation of the system multiplied by dy/ds Taking into account the latter expression as well, we obtain

$$\frac{T}{\sqrt{1 - u^2}}\frac{du}{ds} = uF_y(u) - \sqrt{1 - u^2}F_x(u) \tag{2.4}$$

We divide Equation (2.3) by (2.4)

$$\frac{\sqrt{1 - u^2}}{T}\frac{dT}{du} = \frac{uF_x(u) + \sqrt{1 - u^2}F_y(u)}{\sqrt{1 - u^2}F_x(u) - uF_y(u)}$$

Integration of this expression gives us the tension in the string

$$T(u) = C_1^\circ \exp \int \frac{uF_x(u) + \sqrt{1 - u^2}F_y(u)}{(1 - u^2)F_x(u) - u\sqrt{1 - u^2}F_y(u)}\, du$$

We now integrate Equation (2.4) to find

$$s(u)=\int\frac{T(u)\,du}{u\sqrt{1-u^2F_y(u)+(u^2-1)F_x(u)}}+C_2^\circ$$

Let us now find the present ooordinates of the string as functions of the direction cosine. We proceed by integrating Equations (2.2) and making use of the latter expression. This gives us

$$x(u)=\int\frac{uT(u)\,du}{u\sqrt{1-u^2F_y(u)+(u^2-1)F_x(u)}}+C_3^\circ$$

$$y(u)=\int\frac{T(u)\,du}{uF_y(u)-\sqrt{1-u^2F_x(u)}}+C_4^\circ$$

The constants C_1°, C_2°, C_3°, C_4°, may be determined from the initial conditions.

Thus, the tension and equilibrium shape of the string are found in quadratures.

3. Let us consider the case where the string is acted upon by a homogeneous force field (e.g. a gravitational field) in addition to the external forces that depend on the orientation of the string.

In constructing the equilibrium equations for the flexible string, we choose a coordinate system such that the direction of one of its axes, e.g. the y-axis, coincides with the direction of the homogeneous field vector. Now, in order to extend the solutions of Sections 1 and 2 to the case in hand, it is sufficient to replace F_y by F_y+q in the final formulas of these solutions (q = const is the absolute value of the homogeneous field force per unit string length).

BIBLIOGRAPHY

1. Popov, A.F., Analiticheskoe reshenie zadachi (An analytic solution of a problem). Zapiski Imperatorskoi Akademii Nauk (Papers of the Imperial Academy of Sciences), Vol.11, Book 1, 1867.
2. Krylov, A.N., O ravnovesii sharovoi miny na techenii (On the equilibrium of a spherical mine in a flow). Izvestiia po minnomu delu (News of mine engineering), № 44, 1909.
3. Kochin, N.E., Ob izgibe trosa zmeikovogo aerostata pod deistviem vetra (On the bending of a kite balloon cable in a wind). *PMM* Vol.10, 1946.
4. Minakov, A.P., Osnovy mekhaniki niti (Principles of the Mechanics of a String). Nauchn.-issled.tr.Mosk.text.in-ta (Scientific Research Transactions of the Moscow Textile Institute), Vol.9, 1941.

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